INVERSE TRIGNOMETRY

f(x)	Domain	Range
$\sin^{-1}x$	[-1,1]	$\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$
$\cos^{-1}x$	[-1,1]	$[0,\pi]$
$\tan^{-1}x$	R	$\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$
$\text{cosec}^{-1}x$	R-[-1,1]	$\left[-\frac{\pi}{2},\frac{\pi}{2}\right]-\{0\}$
$\sec^{-1}x$	R-[-1,1]	$[0,\pi]-\left\{\frac{\pi}{2}\right\}$
$\cot^{-1}x$	R	$(0,\pi)$

f(x)	Increasing/Decreasing	odd/even
$\sin^{-1}x$	Increasing	Odd
$\cos^{-1}x$	Increasing	None
$\tan^{-1}x$	Increasing	Odd
$\text{cosec}^{-1}x$	Decreasing in $(-\infty,-1]$ & $[1,\infty)$	Odd
$\sec^{-1}x$	Increasing in $(-\infty,-1]$ & $[1,\infty)$	None
$\cot^{-1}x$	Decreasing	None

Important points to note

- All Inverse Trigonometric functions are Bounded, Continuous and Aperiodic.
- All the inverse trigonometric functions represent an angle.
- If $x > 0$, then all six inverse trigonometric functions represents angle from 1st quadrant.
- If $x < 0$, then $\sin^{-1}x$, $\tan^{-1}x$ & $\operatorname{cosec}^{-1}x$ represents angle from 4th quadrant.
- If $x < 0$, then $\cos^{-1}x$, $\sec^{-1}x$ & $\cot^{-1}x$ represents angle from 4th quadrant.
- IIIrd quadrant is never used as a range in inverse trigno.

Properties of Inverse trigonometric functions

- **Negative Arguments**

(a) $\sin^{-1}(-x) = -\sin^{-1}x, \ \forall x \in [-1,1]$

(b) $\cos^{-1}(-x) = \pi - \cos^{-1}x, \ \forall x \in [-1,1]$

(c) $\tan^{-1}(-x) = -\tan^{-1}x, \forall x \in R$

(d) $\cot^{-1}(-x) = \pi - \cot^{-1}x, \forall x \in R$

(e) $\sec^{-1}(-x) = \pi - \sec^{-1}x, \forall x \in (-\infty, -1] \cup [1, \infty)$

(f) $\operatorname{cosec}^{-1}(-x) = -\operatorname{cosec}^{-1}x \forall x \in (-\infty, -1) \cup [1, \infty)$

Forward Inverse Identities

Identity	Domain	Range
$y = \sin(\sin^{-1}x) = x$	$[\ 1,1]$	$[\ 1,1]$
$y = \cos(\cos^{-1}x) = x$	$[\ 1,1]$	$[\ 1,1]$
$y = \tan(\tan^{-1}x) = x$	R	R
$y = \cot(\cot^{-1}x) = x$	R	R
$y = \text{cosec}(\text{cosec}^{-1}x) = x$	$\lvert x\rvert \geqq 1$	$\lvert y\rvert \geqq 1$
$y = \sec(\sec^{-1}x) = x$	$\lvert x\rvert \geqq 1$	$\lvert y\rvert \geqq 1$

Inverse Forward Identities

Identity	Domain	Range
$y = \sin^{-1}(\sin x) = x$	R	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$y = \cos^{-1}(\cos x) = x$	R	$[0, \pi]$
$y = \tan^{-1}(\tan x) = x$	$R - \left\{(2n-1)\frac{\pi}{2} n \in I\right\}$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
$y = \tan^{-1}(\tan x) = x$	$R - \{n\pi\}$ $n \in I$	$(0, \pi)$
$y = \text{cosec}^{-1}(\text{cosec}x) = x$	$R - \{n\pi\},\ n \in I$	$\left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$
$y = \sec^{-1}(\sec x) = x$	$R - \left\{(2n-1)\frac{\pi}{2} n \in I\right\}$	$\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$

Note :

- $\sin^{-1}(\sin x)$, $\cos^{-1}(\cos x)$, $\text{cosec}^{-1}(\text{cosec}\,x)$, $\sec^{-1}(\sec x)$ are periodic with 2π
- $\tan^{-1}(\tan x)$, $\cot^{-1}(\cot x)$ are periodic with π

Reciprocal Arguments

$\text{cosec}^{-1}x = \sin^{-1}\frac{1}{x}; |x| \geq 1$ & $\sin^{-1}x = \text{cosec}^{-1}\frac{1}{x}; |x| \leq 1$, $x \neq 0$

$\sec^{-1}x = \cos^{-1}\frac{1}{x}; |x| \geq 1$ & $\cos^{-1}x = \sec^{-1}\frac{1}{x}; |x| \leq 1$

$\cot^{-1}x = \tan^{-1}\left(\frac{1}{x}\right), \; x \in (0,\infty) = \pi + \tan^{-1}\left(\frac{1}{x}\right), \; x \in (-\infty, 0)$

Sum of Angles

- $\sin^{-1}x + \cos^{-1}x = \frac{\pi}{2}, \quad \forall\, x \in [-1, 1]$
- $\tan^{-1}x + \cot^{-1}x = \frac{\pi}{2}, \quad \forall\, x \in R$
- $\sec^{-1}x + \text{cosec}^{-1}x = \frac{\pi}{2}, \quad \forall\, x \in (-\infty, -1] \cup [1, \infty)$

Converting ITF into other ITF

$$\sin^{-1}x = \cos^{-1}\sqrt{1-x^2} = \tan^{-1}\frac{x}{\sqrt{1-x^2}} = \cot^{-1}\frac{\sqrt{1-x^2}}{x}$$

$$= \sec^{-1}\frac{1}{\sqrt{1-x^2}} = \mathrm{cosec}^{-1}\frac{1}{x},\ 0 \le x \le 1$$

$$\cos^{-1}x = \sin^{-1}\sqrt{1-x^2} = \tan^{-1}\frac{\sqrt{1-x^2}}{x} = \cot^{-1}\frac{x}{\sqrt{1-x^2}}$$

$$= \sec^{-1}\frac{1}{x} = \mathrm{cosec}^{-1}\frac{1}{\sqrt{1-x^2}},\ 0 \le x \le 1$$

$$\tan^{-1}x = \sin^{-1}\frac{x}{\sqrt{1+x^2}} = \cos^{-1}\frac{1}{\sqrt{1+x^2}} = \cot^{-1}\frac{1}{x}$$

$$= \sec^{-1}\sqrt{1+x^2} = \mathrm{cosec}^{-1}\frac{\sqrt{1+x^2}}{x},\ x \ge 0$$

Other Formulas if x & y > 0

- $\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$ $\quad xy < 1$
- $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$ $\quad xy > -1$
- $\tan^{-1}x + \tan^{-1}y = \pi + \tan^{-1}\left(\frac{x+y}{1-xy}\right)$ $\quad xy > 1$

$$\tan^{-1}x+\tan^{-1}y+\tan^{-1}z=\tan^{-1}\left(\frac{x+y+z-xyz}{1-xy-yz-zx}\right)$$

Other Formula's

When, $x^2+y^2\le 1$

$$\sin^{-1}x\pm\sin^{-1}y=\sin^{-1}\left[x\sqrt{1-y^2}\pm y\sqrt{1-x^2}\right]$$

$$\cos^{-1}x\pm\cos^{-1}y=\cos^{-1}\left[xy\mp\sqrt{1-x^2}\sqrt{1-y^2}\right]$$

When, $x^2+y^2>1$

$$\sin^{-1}x\pm\sin^{-1}y=\pi-\sin^{-1}\left[x\sqrt{1-y^2}\pm y\sqrt{1-x^2}\right]$$

$$\cos^{-1}x\pm\cos^{-1}y=\pi-\cos^{-1}\left[xy\mp\sqrt{1-x^2}\sqrt{1-y^2}\right]$$

Inverse trigonometric ratios of multiple angles

$$2\tan^{-1}x=\tan^{-1}\left(\frac{2x}{1-x^2}\right)=\sin^{-1}\left(\frac{2x}{1+x^2}\right)=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$

$$2\sin^{-1}x=\sin^{-1}(2x\sqrt{1-x^2}),\ \text{if}\ -1\le x\le 1$$

$$2\cos^{-1}x=\cos^{-1}(2x^2-1),\ \text{if}\ -1\le x\le 1$$

$$3\sin^{-1}x=\sin^{-1}(3x-4x^3)\qquad 3\cos^{-1}x=\cos^{-1}(4x^3-3x)$$

$$3\tan^{-1}x=\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$$